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ADDENDUM TO "SEMIPRIME GOLDIE CENTRALIZERS"

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ABSTRACT

This paper gives a full proof of a generalization to theorem 1 in *Semiprime Goldie centralizers* [2]. The original proof was based on prop. 2 in [8], but this proposition turns out to be incomplete. We prove the following theorem: Let G be a finite group of automorphisms acting on a $|G|$-torsion free ring R. Then R^G is Goldie if and only if R is Goldie, and then $Q(R)^{\hat{G}} = Q(R^G)$ and $\dim R \geqq \dim R^G$.

The proof of theorem 1 in *Semiprime Goldie centralizers* [2] was not complete, since it is not generally true that any nonzero divisor of R is a left nonzero divisor of Q (as claimed in [8], p. 110). We will show, however, that T—the nonzero divisors of R^G—stay nonzero divisors of Q and thus that the proof was correct.

We will actually prove a generalization of theorem 1 in [2] to finite groups of automorphisms and also give a converse to it. The main result is:

THEOREM 1. *Let G be a finite group of automorphisms acting on a semiprime $|G|$-torsion free ring R. Then R^G is Goldie if and only if R is Goldie, and then $Q(R)^{\hat{G}} = Q(R^G)$ and* $\dim R \geqq \dim R^G$, *where $\hat{G}$ is an extension of G to $Q(R)$, the full quotient ring of R, and* $\dim R$ *denotes the length of a maximal direct sum of right ideals of R.*

The question of whether R^G is necessarily Goldie if R is semiprime Goldie, which is proved in the theorem, was proved already in 1972 by Carl Faith [4] in the special case when R is an Ore domain. The search for an answer to this question motivated Bergman and Isaacs [1] to prove Proposition 2.3, which is much used in our paper.

Notations and definitions are as in [2].

We also mention the important variant to a result of Bergman and Isaacs [1] which gives a connection between R^G and R.

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LEMMA 1. *Let G be a finite group of automorphisms acting on a $|G|$-torsion free ring R. Let $S \neq 0$ be a subring of R invariant under G. Then $S \cap R^G \neq 0$ or S is nilpotent.*

PROOF. This follows from [1, prop. 2.3].

In the following, G *will denote a finite group of automorphisms acting on a* $|G|$*-torsion free ring* R.

Proposition 1 in [2] is valid for R and G as above, the proof being identical to the proof in [2] where, instead of $J = \cap \phi^k(I)$, we take $J = \bigcap_{g \in G} g(I)$.

Instead of proposition 2 in [2] we use the following generalization, which was proved in [3] but can also be shown to be a consequence of Proposition 1:

PROPOSITION 2. *If R is semiprime and R^G is semisimple Artinian, then R is semisimple Artinian.*

The next proposition gives the converse of the main theorem.

PROPOSITION 3. *If R is semiprime and R is finite dimensional, then R^G is finite dimensional and* $\dim R \geqq \dim R^G$.

PROOF. Semiprimeness of R implies semiprimeness of R^G, by [9]; hence it is enough to show that if $x_1R^G \oplus \cdots \oplus x_mR^G$ is direct in R^G then $x_1R + \cdots + x_mR$ is direct in R, $x_i \in R^G$. If $0 \neq x_jR \cap (\Sigma_{i \neq j} x_iR)$, then, being invariant under G, we get by Lemma 1 and semiprimeness of R that $0 \neq x_jR \cap (\Sigma_{i \neq j} x_iR) \cap R^G$, but if $0 \neq x_jr_j = \Sigma_{i \neq j} x_ir_i \in R^G$, then

$$0 \neq |G| x_jr_j = x_j \sum_{g \in G} g(r_j) = \sum_{i \neq j} x_i \left(\sum_{g \in G} g(r_i) \right) \in x_jR^G \cap \left(\sum_{i \neq j} x_iR^G \right) = 0,$$

a contradiction. Thus Σx_iR is direct.

In the following proposition we deal with a ring R with singular ideal 0 and maximal quotient ring Q (in [2], Q is called a regular right quotient ring of R). We mention that for any $q \in Q$, $q^{-1}R = \{r \in R \mid qr \in R\}$ is essential and $q: q^{-1}R \to R$ is a right R-module homomorphism.

We show next that if G is a group of automorphisms acting on R, then G can be extended to a group of automorphisms $\hat{G}$ acting on Q such that the restriction of $\hat{G}$ to R is G. Categorically, we show that $\operatorname{Aut} R$ is a section of $\operatorname{Aut} Q$.

PROPOSITION 4. *Let R be a ring with singular ideal 0 and maximal quotient ring Q. Then the following hold:*

1) *If* H_1, $H_2 \in \operatorname{Aut} Q$ *and if* $H_1(r) = H_2(r)$ *for all r in R, then* $H_1 = H_2$.

2) *Define* μ *on* Aut R *via*: $\mu(h)(q) = hqh^{-1}$ *for* $h \in \operatorname{Aut} R$ *and* $q \in Q$. *Denote by* $(\operatorname{Aut} Q)_R$ *the group of automorphisms on Q which leave R invariant. Then, the restriction of* $\mu(h)$ *to R is h, so,* $\mu : \operatorname{Aut} R \to (\operatorname{Aut} Q)_R$. *In addition* μ *is a bijective group homomorphism. Hence* $\operatorname{Aut} R \cong (\operatorname{Aut} Q)_R$.

3) *If G is a subgroup of* Aut R, *then* $\hat{G} = \mu(G)$ *is a subgroup of* $(\operatorname{Aut} Q)_R$, *and the restriction of* $\hat{G}$ *to R is G. Hence, in particular,* $R^{\hat{G}} = R^G$.

PROOF. To prove 1), it is enough to show that if $H(r) = r$ for all $r \in R$, then $H(q) = q$ for all $q \in Q$. Let $q \in Q$. Then $(H(q) - q)(q^{-1}R) = 0$, since $H(r) = r$ for all $r \in R$. Thus it follows easily that $H(q) - q = 0$.

Next let us prove 2). First we show that $\mu(h)(q) \in Q$, that is, we show that the domain of the function $\mu(h)(q)$ is an essential right ideal and that $\mu(h)(q)$ is a right R-module homomorphism. The domain of $\mu(h)(q) = hqh^{-1}$ is $h(q^{-1}R)$, which is essential since $q^{-1}R$ is essential and h is an automorphism. To show that hqh^{-1} is a right R-module homomorphism, let $x = h(y) \in h(q^{-1}R)$ and $r \in R$. Then

$$hqh^{-1}(xr) = hq[h^{-1}(h(y))h^{-1}(r)] = hq[yh^{-1}(r)].$$

Since $y \in q^{-1}R$ and q is a right R-module homomorphism, we have

$$hqh^{-1}(xr) = h(q(y)h^{-1}(r)) = (hqh^{-1})(x)r.$$

We have seen so far that $\mu(h)$: $Q \to Q$. The proof that $\mu(h) \in \operatorname{Aut} Q$ and that μ is a group homomorphism is straightforward. Next we show that $\mu(h)(r) = h(r)$, thus $\mu(h) \in (\operatorname{Aut} Q)_R$. Let $r \in R$. Then $R \subset Q$ via $r \to L_r$ (i.e., left multiplication by r); hence $\mu(h)(r) = \mu(h)(L_r) = hL_rh^{-1}$. But $(hL_rh^{-1})(x) = h(rh^{-1}(x)) = L_{h(r)}(x)$ for each $x \in R$. Hence $hL_rh^{-1} = L_{h(r)}$, and we have shown that $\mu(h)(r) = h(r)$.

To see that μ is injective, assume $\mu(h) = id_Q$; then $\mu(h)(r) = r$ for all $r \in R$. But by the above $\mu(h)(r) = h(r)$, so $h(r) = r$ for all $r \in R$, that is, $h = id_R$. Finally, μ is onto $(\operatorname{Aut} Q)_R$, since if $H \in (\operatorname{Aut} Q)_R$ and if h is the restriction of H to R, then $\mu(h)(r) = H(r)$ for all $r \in R$; hence by 1), $\mu(h) = H$.

The proof of 3) follows from 2).

We are ready to prove the main theorem, part of which was shown independently by V. K. Harchenko [5].

THEOREM 1. *Let G be a finite group of automorphisms acting on a semiprime* $|G|$*-torsion free ring R. Then* R^G *is Goldie if and only if R is Goldie, and then* $Q(R)^{\hat{G}} = Q(R^G)$ *and* $\dim R \geqq \dim R^G$.

PROOF. If R is semiprime Goldie, then by Proposition 3 R^G is finite dimensional and $\dim R \geqq \dim R^G$. Since R^G is a subring of R, it inherits the maximum condition on right annihilators; hence R^G is Goldie.

Now assume R^G is Goldie. Since R is a semiprime $|G|$-torsion free ring, we have by [8, cor. 5] that R^G is semiprime; thus R^G is semiprime Goldie.

If we now repeat the proof of theorem 1 in [2], and denote by T the set of nonzero divisors of R^G, then all that remains to be shown in order to prove that T is a right denominator set for R is that, for any $t \in T$, t is a left nonzero divisor in Q. To do this, let us extend G to Q as suggested in Proposition 4. Since $|\hat{G}| = |G|$, and since for each $q \in Q$, $qR \cap R \neq 0$, it follows that Q is $|\hat{G}|$-torsion free, and, since R is semiprime, that Q is semiprime.

Now let $t \in T$, and assume that $l(t) = \{q \in Q \mid qt = 0\} \neq 0$. Since $l(t)$ is invariant under $\hat{G}$, an application of Lemma 1 to $\hat{G}$ and Q gives $l(t) \cap Q^{\hat{G}} \neq 0$.

We shall prove next that for any $q \in Q^{\hat{G}}$, we have $qR^G \cap R^G \neq 0$. Since $qR \cap R \neq 0$ is invariant under $\hat{G}$, Lemma 1 yields $(qR \cap R) \cap Q^{\hat{G}} \neq 0$. But since $qR \cap R \subset R$ and $R^{\hat{G}} = R^G$, we actually have $(qR \cap R) \cap R^G \neq 0$. Hence there are x and y in R such that $0 \neq qx = y \in R^G$, so

$$0 \neq |G| y = \sum_{h \in G} h(y) = \sum_{\hat{h} \in \hat{G}} \hat{h}(y) = \sum \hat{h}(qx) = q \sum \hat{h}(x) = q \sum h(x) \in qR^G \cap R^G.$$

In conclusion, there exists an element q such that $qt = 0$. On the other hand, there are $x, y \in R^G$ such that $0 \neq qx = y \in R^G$. Since R^G is semiprime Goldie and t is a nonzero divisor of R^G, there exist $t_1, x_1 \in R^G$, $t_1 \in T$ such that $tx_1 = xt_1$. But then $0 = qtx_1 = qxt_1 = yt_1 \neq 0$, since $y \in R^G$ and t_1 is a nonzero divisor of R^G. We have reached a contradiction, and thus $l(t) = 0$.

As in [2], we have shown that $Q = R_T$. Since $(R_T)^{\hat{G}} = (R^G)_T$, and since $(R^G)_T$ is semisimple Artinian, Proposition 2 implies that R_T is Artinian. Since R is now an order in R_T, R is Goldie. Since $R_T = Q(R)$, the full ring of quotients of R, we derive from the above that $(Q(R))^{\hat{G}} = Q(R^G)$.

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REFERENCES

1. G. M. Bergman and I. M. Isaacs, *Rings with fixed point free group actions*, Proc. London Math. Soc., 3rd series, **27** (1973), 69–87.

2. M. Cohen, *Semiprime Goldie centralizers*, Israel J. Math. **20** (1975), 37–45.

3. M. Cohen and S. Montgomery, *Semisimple Artinian rings of fixed points*, Canad. Math. Bull. **18** (1975), 189–190.

4. C. Faith, *Galois subrings of Ore domains are Ore domains*, Bull. Amer. Math. Soc. **78** (1972) 1077–1080.

5. V. K. Harchenko, *Galois extensions and rings of fractions*, 2nd All-Union Symposium in Ring Algebra and Modules Theory, Kishinyov, 1974.

6. I. N. Herstein and L. Neumann, *Centralizers in rings*, Annali di Mathematica servie Quarta Tomo CIF (1975), 37–44.

7. R. E. Johnson, *The extended centralizer of a ring over a module*, Proc. Amer. Math Soc. **2** (1951), 891–895.

8. J. Lambek, *Lectures on Rings and Modules*, Blaisdell, Waltham, Mass., 1966.

9. S. Montgomery, *The Jacobson radical and fixed rings of automorphisms* (to appear in Comm. Algebra).

10. L. Small, *Orders in Artinian rings*, J. Algebra **4** (1966), 13–41.

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